

Killing Symmetries as Hamiltonian Constraints.

Luca Lusanna

Sezione INFN di Firenze, Via G. Sansone 1

50019 Sesto Fiorentino (FI), Italy

E-mail: lusanna@fi.infn.it

Abstract

The existence of a Killing symmetry in a gauge theory is equivalent to the addition of extra Hamiltonian constraints in its phase space formulation, which imply restrictions both on the Dirac observables (the gauge invariant physical degrees of freedom) and on the gauge freedom.

When there is a time-like Killing vector field only pure gauge electromagnetic fields survive in Maxwell theory in Minkowski space-time, while in ADM canonical gravity in asymptotically Minkowskian space-times only inertial effects without gravitational waves survive.

November 19, 2015

I. INTRODUCTION

Killing symmetries are of basic importance in the search of exact solutions of Einstein's equations. If the metric tensor satisfies $L_X {}^4g_{\mu\nu} = 0$, the vector field $X = \xi^\mu \partial_\mu$ is a Killing vector field (satisfying the 8 Killing equations $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2 {}^4\Gamma_{\mu\nu}^\alpha \xi_\alpha = 0$) and the space-time has a Killing symmetry and privileged 4-coordinates adapted to it leading to a simplification of Einstein equations.

It is known from the work in Refs.[1–5] that for Einstein and Einstein-Maxwell theories ¹ the presence of Killing symmetries, when the tensors over space-time belong to ordinary Sobolev spaces, introduces singularities in the space of 4-metrics. This space is not a manifold but has a *cone over cone* structure of singularities: there is a cone of 4-metrics with a Killing symmetry, from each point of this cone emerges a cone of 4-metrics with two Killing symmetries and so on. Choquet-Bruhat [8, 9] has shown that if the tensors belong to certain weighted Sobolev spaces ² then these singularities disappear, because the implied boundary conditions on the tensors after a 3+1 splitting of the space-time exclude the existence of Killing vectors ³. Therefore if all the fields belong to suitable *weighted Sobolev spaces* then:

- i) the admissible space-like hyper-surfaces (the 3-spaces of a 3+1 splitting) are Riemannian 3-manifolds without asymptotically vanishing Killing vectors [8, 9]; as a consequence of the assumed boundary conditions no Killing vectors can be present except the asymptotic Killing symmetries (the 10 ADM Poincare' generators [11] of the asymptotic Poincare' group) in the case of asymptotically Minkowskian space-times;
- ii) the inclusion of particle physics leads to a formulation without Gribov ambiguity [6, 7].

As a consequence, in this class of space-times the spaces of 4-metrics and of 3-metrics (after a 3+1 splitting) should be smooth manifolds without singularities. If a space-time of this class without non-asymptotic Killing symmetries and with the fields belonging to suitable weighted Sobolev spaces is globally hyperbolic, topologically trivial, asymptotically Minkowskian and without super-translations [12] then there is a well established Hamiltonian description (using Dirac's theory of constraints [13, 14]) of both metric and tetrad

¹ Einstein-Yang-Mills is even more complicated due to the gauge symmetries, i.e. the Gribov ambiguity, of the Yang-Mills connection [6, 7].

² They have been introduced to validate tensor decompositions like the Hodge one at spatial infinity in asymptotically flat space-times. See Ref.[10].

³ Analogously Moncrief [6] has shown that gauge symmetries and Gribov ambiguity are absent in certain weighted Sobolev spaces.

gravity [15–21] (see Refs.[22] for reviews). This is due to the fact that in these space-times one can make a consistent 3+1 splitting with instantaneous 3-spaces (i.e. a clock synchronization convention) centered on a time-like observer used as origin of (world scalar) radar 4-coordinates [23, 24]: in this way the notion of non-inertial frames defined in Minkowski space-time in Refs. [25] can be extended to this class of curved space-times. The absence of super-translations implies that the SPI group [26] of asymptotic Killing symmetries is reduced to the asymptotic ADM Poincaré group [15, 16].

To our knowledge there is no Hamiltonian treatment of Killing symmetries. In this paper we try to see what happens if we relax the hypothesis of weighted Sobolev spaces and we add by hand the ten Killing equations corresponding to a *given* Killing vector field $X = \xi^\mu \partial_\mu$ or $X = \xi^\tau \partial_\tau + \xi^r \partial_r$ (in radar 4-coordinates adapted to the 3+1 splitting). The ten Killing equations are restrictions on the 4-metric forcing it to belong to the singularity cone of metrics with one Killing vector.

To implement this program we have to rewrite the 10 Killing equations for the given X as Hamiltonian Dirac constraints to be added by hand to the set of (8 or 14) first class constraints of metric or tetrad gravity. These extra constraints restrict the constraint manifold in phase space to contain only metrics with the given Killing vector field X and subsequently will reduce the space of solutions of Hamilton and Einstein equations. A consistent theory will emerge if the extra constraints are compatible with the existing ones. To check this point we have to ask for the time-preservation of the extra constraints: their Poisson bracket with the Dirac Hamiltonian ⁴ must vanish.

Since the ten Killing constraints corresponding to a Killing vector must be preserved in time, other 10 constraints emerge (they too should be preserved in time, but this should not add new constraints). Therefore each Killing vector gives rise to twenty extra constraints identifying a well defined sub-manifold of the constraint manifold.

If there are no pathologies, the net effect of the Killing symmetry will be to impose a symmetry pattern on the tidal variables (the physical degrees of freedom of the gravitational

⁴ The extra constraints must not be added to the Dirac Hamiltonian with Dirac multipliers as we do with primary constraints, because they are interpreted as restrictions on the field configurations added by hand. Only if one would find a singular Lagrangian invariant under local gauge transformations generated by a Killing symmetry, one could use the standard Dirac algorithm with all the Killing constraints already present.

field becoming the two polarizations of the gravitational waves in the linearized theory) and on the inertial ones (describing the gauge freedom in the choice of the coordinates). In particular we will show that a time-like Killing vector eliminates completely the tidal degrees of freedom, so that the resulting space-time contains only stationary inertial effects and maybe singularities like black holes. Extra Killing symmetries should only restrict these residual inertial effects.

Before treating gravity, we will show what is the effect of symmetry on the electro-magnetic field in Minkowski space-time. This will be done by adding the requirement that the connection A_μ satisfies the symmetry requirement $L_X A_\mu = 0$ for a given vector field $X = \xi^\mu \partial_\mu$. Now four constraints are associated to each such vector: their time preservation generates other four constraints, so that each symmetry has eight associated constraints. We will see that in the case of a time-like symmetry vector field only *pure gauge* electro-magnetic potentials survive: there are no radiative degrees of freedom (eventually a singularity in the case of magnetic monopoles could be present). With other types of symmetry vector fields the radiative degrees of freedom (and also the gauge freedom) are forced to obey the symmetry.

In Section II we describe the 3+1 splittings of the quoted space-times and the notion of radar 4-coordinates, which allows to give a description of gravity in terms of world-scalar quantities in non-inertial frames.

In Section III, after remembering the formulation [25] of the Hamiltonian version of electro-magnetism in non-inertial frames and in the inertial rest frame in Minkowski space-time, we study the Hamiltonian version of symmetry conditions (special relativistic Killing symmetries) on the electro-magnetic field. The case of a time-like symmetry is treated explicitly.

In Section IV, after remembering the formulation [20] of Hamiltonian tetrad gravity in the non-inertial frames of the quoted space-times, we give the Hamiltonian formulation of Killing symmetries in the York canonical basis of tetrad gravity and we study the implications of the presence of a time-like Killing vector.

In Appendix A we give the Hamiltonian expression of the extrinsic curvature and of the Christoffel symbols.

II. 3+1 SPLITTING AND RADAR 4-COORDINATES

Assume that the world-line $x^\mu(\tau)$ of an arbitrary time-like observer carrying a standard atomic clock is given either in Minkowski space-time or in the quoted class of Einstein space-times: τ is an arbitrary monotonically increasing function of the proper time of this clock. Then one gives an admissible 3+1 splitting of the asymptotically flat space-time, namely a nice foliation with space-like instantaneous 3-spaces Σ_τ . It is the mathematical idealization of a protocol for clock synchronization: all the clocks in the points of Σ_τ sign the same time of the atomic clock of the observer. The observer and the foliation define a global non-inertial reference frame after a choice of 4-coordinates. On each 3-space Σ_τ one chooses curvilinear 3-coordinates σ^r having the observer as origin. The quantities $\sigma^A = (\tau; \sigma^r)$ are the either Lorentz- or world-scalar and observer-dependent *radar 4-coordinates*, first introduced by Bondi [23, 24].

If $x^\mu \mapsto \sigma^A(x)$ is the coordinate transformation from world 4-coordinates x^μ having the observer as origin to radar 4-coordinates, its inverse $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the *embedding* functions $z^\mu(\tau, \sigma^r)$ describing the 3-spaces Σ_τ as embedded 3-manifolds into the asymptotically flat space-time. Let $z^\mu_A(\tau, \sigma^u) = \partial z^\mu(\tau, \sigma^u) / \partial \sigma^A$ denote the gradients of the embedding functions with respect to the radar 4-coordinates. The space-like 4-vectors $z^\mu_r(\tau, \sigma^u)$ are tangent to Σ_τ , so that the unit time-like normal $l^\mu(\tau, \sigma^u)$ is proportional to $\epsilon^\mu_{\alpha\beta\gamma} [z^\alpha_1 z^\beta_2 z^\gamma_3](\tau, \sigma^u)$ ($\epsilon_{\mu\alpha\beta\gamma}$ is the Levi-Civita tensor). Instead $z^\mu_\tau(\tau, \sigma^u)$ is a time-like 4-vector skew with respect to the 3-spaces leaves of the foliation. In special relativity (SR), see Refs. [25], one has $z^\mu_\tau(\tau, \sigma^r) = [N l^\mu + N^r z^\mu_r](\tau, \sigma^r)$ with $N(\tau, \sigma^r) = \epsilon [z^\mu_\tau l_\mu](\tau, \sigma^r) = 1 + n(\tau, \sigma^r) > 0$ and $N_r(\tau, \sigma^r) = -\epsilon [z^\mu_\tau \eta_{\mu\nu} z^\nu_r](\tau, \sigma^r)$ being the lapse and shift functions respectively of the *global non-inertial frame* of Minkowski space-time so defined.

In SR the classical fields, for instance the Klein-Gordon field $\tilde{\phi}(x^\mu)$, have to be replaced with fields knowing the 3+1 splitting: $\phi(\tau, \sigma^r) = \tilde{\phi}(z^\mu(\tau, \sigma^r))$. With parametrized Minkowski theories [25, 27], one can give a Lagrangian formulation of classical fields in non-inertial frames with a Lagrangian depending also on the embedding variables $z^\mu(\tau, \sigma^r)$. The resulting action is invariant under frame preserving diffeomorphisms. As a consequence the embedding variables are gauge variables and the transition from a non-inertial frame to another (either non-inertial or inertial) one is a gauge transformation. Inertial frames are a special case of this description. For every isolated system with a conserved time-like total 4-momentum P^μ one can define the *inertial rest frame* as the one in which the Euclidean 3-spaces are

orthogonal to P^μ .

In general relativity (GR) the dynamical fields are the components ${}^4g_{\mu\nu}(x)$ of the 4-metric and not the embeddings $x^\mu = z^\mu(\tau, \sigma^r)$ defining the admissible 3+1 splittings of space-time. Now the gradients $z_A^\mu(\tau, \sigma^r)$ of the embeddings give the transition coefficients from radar to world 4-coordinates, so that the components ${}^4g_{AB}(\tau, \sigma^r) = z_A^\mu(\tau, \sigma^r) z_B^\nu(\tau, \sigma^r) {}^4g_{\mu\nu}(z(\tau, \sigma^r))$ of the 4-metric will be the dynamical fields in the ADM action [11].

Let us remark that *the ten quantities ${}^4g_{AB}(\tau, \sigma^r)$ are 4-scalars of the space-time due to the use of the world-scalar radar 4-coordinates.* In each 3-space Σ_τ considered as a 3-manifold with 3-coordinates σ^r (and not as a 3-sub-manifold of the space-time) ${}^4g_{\tau r}(\tau, \sigma^u)$ is a 3-vector and ${}^4g_{rs}(\tau, \sigma^u)$ is a 3-tensor. Therefore *all the components of "radar tensors", i.e. tensors expressed in radar 4-coordinates, are 4-scalars of the space-time* [21].

In the considered class of Einstein space-times the ten *strong* asymptotic ADM Poincaré generators P_{ADM}^A , J_{ADM}^{AB} (they are fluxes through a 2-surface at spatial infinity) are well defined functionals of the 4-metric fixed by the boundary conditions at spatial infinity and of matter (when present). These ten strong generators can be expressed [15, 16] in terms of the weak asymptotic ADM Poincaré generators (integrals on the 3-space of suitable densities) plus first class constraints. The absence of super-translations implies that the ADM 4-momentum is asymptotically orthogonal to the instantaneous 3-spaces (they tend to a Euclidean 3-space at spatial infinity). As a consequence each 3-space of the global non-inertial frame is a *non-inertial rest frame* of the 3-universe. At spatial infinity there are asymptotic inertial observers carrying a flat tetrad whose spatial axes are identified by the fixed stars of star catalogues.

The 4-metric ${}^4g_{AB}$ has signature $\epsilon(+---)$ with $\epsilon = \pm$ (the particle physics, $\epsilon = +$, and general relativity, $\epsilon = -$, conventions). Flat indices (α) , $\alpha = o, a$, are raised and lowered by the flat Minkowski metric ${}^4\eta_{(\alpha)(\beta)} = \epsilon(+---)$. We define ${}^4\eta_{(a)(b)} = -\epsilon\delta_{(a)(b)}$ with a positive-definite Euclidean 3-metric. From now on we shall denote the curvilinear 3-coordinates σ^r with the notation $\vec{\sigma}$ for the sake of simplicity. Usually the convention of sum over repeated indices is used, except when there are too many summations. The symbol \approx means Dirac weak equality, while the symbol $\stackrel{\circ}{=}$ means evaluated by using the equations of motion.

III. ELECTRO-MAGNETISM

In SR the non-dynamical Minkowski space-time admits 10 Killing vectors, the generators of the algebra of the kinematical Poincare' group connecting inertial frames. Let us see what happens to the physical degrees of freedom, the Dirac observables (DO), of the electro-magnetic field in absence of matter, when the electro-magnetic potential is assumed to be left invariant, i.e. $\mathcal{L}_X A = 0$, by anyone of the 10 Poincare' generators. If matter is present, also the matter fields must be assumed to be invariant under the action of the Killing vector field X . In other words, let us look at what kind of restrictions on the phase space of the electro-magnetic field are implied by anyone of these Killing symmetries. This will be done in the inertial rest frame of the electro-magnetic field after a review of its Hamiltonian formulation..

A. Canonical Basis for Electro-Magnetism in the Inertial Rest Frame of Minkowski Space-Time

Let us consider Dirac's Hamiltonian formulation [28] of the electro-magnetic field, reformulated [25] in the rest-frame instant form of dynamics on the Wigner 3-space Σ_τ , i.e. on the space-like hyper-plane orthogonal to the conserved 4-momentum of the isolated system formed by the electro-magnetic field (P^μ is the conserved 4-momentum of the field configuration). This is an inertial frame centered on the covariant Fokker-Pryce center of inertia and uses radar 4-coordinates $(\tau, \vec{\sigma})$. In this formulation there is no breaking of covariance, since all the quantities on a Wigner 3-space are either Lorentz scalars or Wigner spin 1 3-vectors. The Wigner 3-space Σ_τ at time τ is the intrinsic rest frame of the isolated system at time τ . With respect to an arbitrary inertial frame the Wigner hyper-planes are described by the following embedding

$$z^\mu(\tau, \vec{\sigma}) = x_s^\mu(\tau) + \epsilon_r^\mu(u(P))\sigma^r, \quad (3.1)$$

with $x_s^\mu(\tau) = x^\mu(0) + u^\mu(P)\tau$ being the world-line of the Fokker-Pryce inertial observer. The space-like 4-vectors $\epsilon_r^\mu(u(P))$ together with the time-like one $\epsilon_o^\mu(u(P))$ are the columns of the standard Wigner boost for time-like Poincare' orbits that sends the time-like four-vector P^μ to its rest-frame form $\overset{\circ}{P}^\mu = \sqrt{P^2}(1; \vec{0})$: $\epsilon_o^\mu(u(P)) = u^\mu(P) = P^\mu/\sqrt{P^2}$, $\epsilon_r^\mu(u(P)) = \left(-u_r(P); \delta_r^i - \frac{u^i(P)u_r(P)}{1+u^o(P)}\right)$.

The configuration variable is the Lorentz-scalar electro-magnetic potential $A_A(\tau, \vec{\sigma}) =$

$z_A^\mu(\tau, \vec{\sigma}) \tilde{A}_\mu(z^\beta(\tau, \vec{\sigma}))$, whose associated field strength is $F_{AB}(\tau, \vec{\sigma}) = \partial_A A_B(\tau, \vec{\sigma}) - \partial_B A_A(\tau, \vec{\sigma}) = z_A^\mu(\tau, \vec{\sigma}) z_B^\nu(\tau, \vec{\sigma}) \tilde{F}_{\mu\nu}(z^\beta(\tau, \vec{\sigma}))$. The conjugate momentum variables are a scalar $\pi^\tau(\tau, \vec{\sigma}) \approx 0$ and a Wigner 3-vector $\pi^r(\tau, \vec{\sigma}) = E^r(\tau, \vec{\sigma})$. $E^r(\tau, \vec{\sigma})$ and $B^r(\tau, \vec{\sigma})$ are the components of the electric and magnetic fields in the inertial rest frame.

The gauge degrees of freedom (A_τ, η) have been separated from the transverse DO's $(A_{\perp r}, \pi_\perp^r = E_\perp^r)$ ($\vec{\partial} \cdot \vec{A}_\perp = \vec{\partial} \cdot \vec{\pi}_\perp = 0$) by means of a Shanmughadhasan canonical transformation [29, 30] adapted to the two scalar first class constraints ($\pi^\tau(\tau, \vec{\sigma}) \approx 0$ and the Gauss law $\Gamma(\tau, \vec{\sigma}) = \vec{\partial} \cdot \vec{\pi}(\tau, \vec{\sigma}) \approx 0$) generators of the Hamiltonian electro-magnetic gauge transformations ($\Delta = -\vec{\partial}_\sigma^2$, $\square = \partial_\tau^2 + \Delta$)

$$\begin{array}{|c|} \hline A_A \\ \hline \pi^A \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline A_\tau & \eta & A_{\perp r} \\ \hline \pi^\tau \approx 0 & \Gamma \approx 0 & \pi_\perp^r \\ \hline \end{array}$$

$$A_r(\tau, \vec{\sigma}) = \partial_r \eta(\tau, \vec{\sigma}) + A_\perp^r(\tau, \vec{\sigma}), \quad \pi^r(\tau, \vec{\sigma}) = \pi_\perp^r(\tau, \vec{\sigma}) + \frac{1}{\Delta_\sigma} \frac{\partial}{\partial \sigma^r} \Gamma(\tau, \vec{\sigma}),$$

$$\eta(\tau, \vec{\sigma}) = -\frac{1}{\Delta_\sigma} \frac{\partial}{\partial \vec{\sigma}} \cdot \vec{A}(\tau, \vec{\sigma}),$$

$$A_\perp^r(\tau, \vec{\sigma}) = (\delta^{rs} + \frac{\partial_\sigma^r \partial_\sigma^s}{\Delta_\sigma}) A_s(\tau, \vec{\sigma}), \quad \pi_\perp^r(\tau, \vec{\sigma}) = (\delta^{rs} + \frac{\partial_\sigma^r \partial_\sigma^s}{\Delta_\sigma}) \pi_s(\tau, \vec{\sigma}),$$

$$\{A_\tau(\tau, \vec{\sigma}), \pi^\tau(\tau, \vec{\sigma}')\} = -\{\eta(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}')\} = \delta^3(\vec{\sigma} - \vec{\sigma}'),$$

$$\{A_\perp^r(\tau, \vec{\sigma}), \pi_\perp^s(\tau, \vec{\sigma}')\} = -(\delta^{rs} + \frac{\partial_\sigma^r \partial_\sigma^s}{\Delta_\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}'). \quad (3.2)$$

The Dirac Hamiltonian is $(\lambda_\tau(\tau, \vec{\sigma}))$ is the arbitrary Dirac multiplier associated to the primary constraint $\pi^\tau(\tau, \vec{\sigma}) \approx 0$

$$H_D = H_c + \int d^3\sigma [\lambda_\tau \pi^\tau - A_\tau \Gamma](\tau, \vec{\sigma}), \quad H_c = \frac{1}{2} \int d^3\sigma [\vec{\pi}_\perp^2 + \vec{B}^2](\tau, \vec{\sigma}),$$

\Downarrow *kinematical Hamilton equations*

$$\partial_\tau A_\tau(\tau, \vec{\sigma}) \stackrel{\circ}{=} \lambda_\tau(\tau, \vec{\sigma}), \quad \partial_\tau \eta(\tau, \vec{\sigma}) \stackrel{\circ}{=} A_\tau(\tau, \vec{\sigma}), \quad \partial_\tau A_{\perp r}(\tau, \vec{\sigma}) \stackrel{\circ}{=} -\pi_{\perp r}(\tau, \vec{\sigma}),$$

dynamical Hamilton equations

$$\partial_\tau \pi_{\perp}^r(\tau, \vec{\sigma}) \stackrel{\circ}{=} \triangle A_{\perp}^r(\tau, \vec{\sigma}), \quad \Rightarrow \quad \square A_{\perp r}(\tau, \vec{\sigma}) \stackrel{\circ}{=} 0. \quad (3.3)$$

To fix the gauge we must only add a gauge fixing $\varphi_\eta(\tau, \vec{\sigma}) \approx 0$ to the Gauss law, which determines η . Its time constancy, i.e. $\partial_\tau \varphi_\eta(\tau, \vec{\sigma}) + \{\varphi_\eta(\tau, \vec{\sigma}), H_D\} = \varphi_{A_\tau}(\tau, \vec{\sigma}) \approx 0$, will generate the gauge fixing $\varphi_{A_\tau}(\tau, \vec{\sigma}) \approx 0$ for A_τ . Finally the time constancy $\partial_\tau \varphi_{A_\tau}(\tau, \vec{\sigma}) + \{\varphi_{A_\tau}(\tau, \vec{\sigma}), H_D\} \approx 0$ will determine the Dirac multiplier $\lambda_\tau(\tau, \vec{\sigma})$. By adding these *two* gauge fixing constraints to the first class constraints $\pi^\tau(\tau, \vec{\sigma}) \approx 0$, $\Gamma(\tau, \vec{\sigma}) \approx 0$, one gets two pairs of second class constraints allowing the elimination of the gauge degrees of freedom so that only the DO's survive.

B. Special Relativistic Killing Symmetries of the Electro-Magnetic Field

Given a vector field

$$X = \xi^A(\tau, \vec{\sigma}) \partial_A = \xi^\tau(\tau, \vec{\sigma}) \partial_\tau + \xi^r(\tau, \vec{\sigma}) \partial_r, \quad (3.4)$$

let us look for electro-magnetic potentials $A_A(\tau, \vec{\sigma})$ satisfying the Killing equations

$$\varphi_A(\tau, \vec{\sigma}) = [L_X A]_A(\tau, \vec{\sigma}) = \xi^B(\tau, \vec{\sigma}) \partial_B A_A(\tau, \vec{\sigma}) + A_B(\tau, \vec{\sigma}) \partial_A \xi^B(\tau, \vec{\sigma}) = 0. \quad (3.5)$$

In phase space the four equations $\varphi_A(\tau, \vec{\sigma}) = 0$ must be reinterpreted as *four constraints* added by hand and restricting the configurations of the electro-magnetic field to those having

this Killing symmetry. Since the electro-magnetic gauge theory has only two gauge degrees of freedom, the Killing equations are also a restriction on the DO's $\vec{A}_\perp, \vec{\pi}_\perp$.

Moreover in phase space we have to ask (like for the ordinary gauge fixings) the time constancy of these (added by hand) four extra constraints. Again a priori this will add other four conditions, which have to be studied and again asked to be preserved in time (and so on ..). Only at the end, if this procedure does not lead to inconsistencies, we can say that the theory admits electro-magnetic fields with the given Killing symmetry.

To rewrite the four Killing equations $\varphi_A(\tau, \vec{\sigma}) = 0$ as constraints we must use the first (kinematical) half of Hamilton equations (3.3) to replace the velocities (∂_τ derivatives of the canonical variables) with their phase space expression.

Let us study in detail the four Killing equations (3.5).

1) The $A = \tau$ Killing equation generates the following constraint

$$\begin{aligned} \varphi_\tau(\tau, \vec{\sigma}) &= \left[\xi^\tau \partial_\tau A_\tau + \xi^s \partial_s A_\tau + A_\tau \partial_\tau \xi^\tau + A_s \partial_\tau \xi^s \right](\tau, \vec{\sigma}) \stackrel{\circ}{=} \\ &\stackrel{\circ}{=} \left[\xi^\tau \lambda_\tau + \xi^s \partial_s A_\tau + A_\tau \partial_\tau \xi^\tau + (\partial_s \eta + A_{\perp s}) \partial_\tau \xi^s \right](\tau, \vec{\sigma}) \approx 0, \end{aligned} \quad (3.6)$$

We have

$$\begin{aligned} \{\varphi_\tau(\tau, \vec{\sigma}), H_c\} &= \partial_\tau \xi^s(\tau, \vec{\sigma}) \{A_{\perp s}(\tau, \vec{\sigma}), H_c\} \stackrel{\circ}{=} \partial_\tau \xi^s(\tau, \vec{\sigma}) \pi_{\perp s}(\tau, \vec{\sigma}), \\ \{\varphi_\tau(\tau, \vec{\sigma}), \pi^\tau(\tau, \vec{\sigma}_1)\} &= \xi^s(\tau, \vec{\sigma}) \partial_s \delta^3(\vec{\sigma} - \vec{\sigma}_1) + \delta^3(\vec{\sigma} - \vec{\sigma}_1) \partial_\tau \xi^\tau(\tau, \vec{\sigma}), \\ \{\varphi_\tau(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}_1)\} &= -\partial_\tau \xi^s(\tau, \vec{\sigma}) \partial_s \delta^3(\vec{\sigma} - \vec{\sigma}_1), \\ \{\varphi_\tau(\tau, \vec{\sigma}), \int d^3\sigma_1 [\lambda_\tau \pi^\tau](\tau, \vec{\sigma}_1)\} &= \xi^s(\tau, \vec{\sigma}) \partial_s \lambda_\tau(\tau, \vec{\sigma}), \\ \{\varphi_\tau(\tau, \vec{\sigma}), -\int d^3\sigma_1 [A_\tau \Gamma](\tau, \vec{\sigma}_1)\} &= -\partial_\tau \xi^s(\tau, \vec{\sigma}) \partial_s A_\tau(\tau, \vec{\sigma}), \end{aligned} \quad (3.7)$$

The time constancy of $\varphi_\tau(\tau, \vec{\sigma}) \approx 0$ generates the extra constraint [the Killing constraint has an explicit τ -dependence through the $\xi^A(\tau, \vec{\sigma})$ components of the Killing vector field and the Dirac multiplier $\lambda_\tau(\tau, \vec{\sigma})$]

$$\begin{aligned}
\psi_\tau(\tau, \vec{\sigma}) &= \partial_\tau \varphi_\tau(\tau, \vec{\sigma}) + \{\varphi_\tau(\tau, \vec{\sigma}), H_D\} \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} \left[\partial_\tau \xi^s \pi_{\perp s} + \xi^s \partial_s \lambda_\tau + \partial_\tau \xi^\tau \lambda_\tau + \xi^\tau \partial_\tau \lambda_\tau + \right. \\
&\quad \left. + A_\tau \partial_\tau^2 \xi^\tau + (\partial_s \eta + A_{\perp s}) \partial_\tau^2 \xi^s \right](\tau, \vec{\sigma}) \approx 0.
\end{aligned} \tag{3.8}$$

2) The $A = r$ Killing equations generate the following three constraints

$$\begin{aligned}
\varphi_r(\tau, \vec{\sigma}) &= \left[\xi^\tau \partial_\tau (\partial_r \eta + A_{\perp r}) + \xi^s \partial_s (\partial_r \eta + A_{\perp r}) + \right. \\
&\quad \left. + A_\tau \partial_r \xi^\tau + (\partial_s \eta + A_{\perp s}) \partial_r \xi^s \right](\tau, \vec{\sigma}) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} \left[\xi^\tau (\partial_r A_\tau + \pi_{\perp r}) + \partial_r (\xi^s \partial_s \eta) + A_\tau \partial_r \xi^\tau + \right. \\
&\quad \left. + \xi^s \partial_s A_{\perp r} + A_{\perp s} \partial_r \xi^s \right](\tau, \vec{\sigma}) \approx 0.
\end{aligned} \tag{3.9}$$

We have

$$\begin{aligned}
\{\varphi_r(\tau, \vec{\sigma}), H_c\} &= \left[\xi^\tau \triangle A_{\perp r} + \xi^s \partial_s \pi_{\perp r} + \pi_{\perp s} \partial_r \xi^s \right](\tau, \vec{\sigma}), \\
\{\varphi_r(\tau, \vec{\sigma}), \int d^3 \sigma_1 [\lambda_\tau \pi^\tau](\tau, \vec{\sigma}_1)\} &= \partial_r [\lambda_\tau \xi^\tau](\tau, \vec{\sigma}), \\
\{\varphi_r(\tau, \vec{\sigma}), - \int d^3 \sigma_1 [A_\tau \Gamma](\tau, \vec{\sigma}_1)\} &= -\partial_r [\xi^s \partial_s A_\tau](\tau, \vec{\sigma}).
\end{aligned} \tag{3.10}$$

The time constancy of the Killing constraints $\varphi_r(\tau, \vec{\sigma}) \approx 0$ generates the extra constraints (the dynamical Hamilton equations (3.3) are used)

$$\begin{aligned}
\psi_r(\tau, \vec{\sigma}) &= \partial_\tau \varphi_r(\tau, \vec{\sigma}) + \{\varphi_r(\tau, \vec{\sigma}), H_D\} \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} \left[\partial_\tau \xi^\tau (\partial_r A_\tau + \pi_{\perp r}) + \partial_r (\partial_\tau \xi^s \partial_s \eta) + A_\tau \partial_r \partial_\tau \xi^\tau + \right. \\
&\quad \left. + \partial_\tau \xi^s \partial_s A_{\perp r} + A_{\perp s} \partial_r \partial_\tau \xi^s + \partial_r (\xi^\tau \lambda_\tau - \xi^s \partial_s A_\tau) + \right. \\
&\quad \left. + \xi^\tau \triangle A_{\perp r} + \xi^s \partial_s \pi_{\perp r} + \pi_{\perp s} \partial_r \xi^s \right](\tau, \vec{\sigma}) \approx 0.
\end{aligned} \tag{3.11}$$

Since we have already 8 constraints on the 6 existing variables, i.e. the 2 gauge variables A_τ , η , and the 4 DO's \vec{a}_\perp , $\vec{\pi}_\perp$, the constraints $\psi_A(\tau, \vec{\sigma}) \approx 0$ must be identically conserved: $\partial_\tau \psi_A(\tau, \vec{\sigma}) = 0$.

C. A Time-like Killing Vector

To understand the meaning of these Killing constraints let us consider the *time-translation Killing vector field* $X = \partial_\tau$ with $\xi^\tau(\tau, \vec{\sigma}) = 1$ and $\xi^s(\tau, \vec{\sigma}) = 0$.

In this case Eqs.(3.6) and (3.3) imply

$$\varphi_\tau(\tau, \vec{\sigma}) = \lambda_\tau(\tau, \vec{\sigma}) \stackrel{\circ}{=} \partial_\tau A_\tau(\tau, \vec{\sigma}) \stackrel{\circ}{=} \partial_\tau^2 \eta(\tau, \vec{\sigma}) \approx 0, \quad \psi_\tau(\tau, \vec{\sigma}) = \partial_\tau \lambda_\tau(\tau, \vec{\sigma}) \approx 0$$

$$\varphi_r(\tau, \vec{\sigma}) = [\partial_r A_\tau + \pi_{\perp r}](\tau, \vec{\sigma}) \stackrel{\circ}{=} [\partial_r A_\tau - \partial_\tau A_{\perp r}](\tau, \vec{\sigma}) \approx 0,$$

$$\psi_r(\tau, \vec{\sigma}) = \left[\partial_r \lambda_\tau + \Delta A_{\perp r} \right](\tau, \vec{\sigma}) \approx \Delta A_{\perp r}(\tau, \vec{\sigma}) \approx 0,$$

$$\partial_r \varphi_r(\tau, \vec{\sigma}) \approx -\Delta A_\tau(\tau, \vec{\sigma}) \approx 0, \quad \partial_r \psi_r(\tau, \vec{\sigma}) = -\Delta \lambda_\tau(\tau, \vec{\sigma}) \approx 0,$$

$$\varphi_{\perp r}(\tau, \vec{\sigma}) = \pi_{\perp r}(\tau, \vec{\sigma}) \approx 0, \quad \psi_{\perp r}(\tau, \vec{\sigma}) = \Delta A_{\perp r}(\tau, \vec{\sigma}) \approx 0. \quad (3.12)$$

Therefore $\psi_\tau(\tau, \vec{\sigma}) \approx 0$ and $\partial_r \psi_r(\tau, \vec{\sigma}) \approx 0$ do not imply constraints being identically satisfied.

The two Killing constraints $\varphi_\tau(\tau, \vec{\sigma}) \approx 0$ and $\partial_r \varphi_r(\tau, \vec{\sigma}) \approx 0$ are two gauge fixing constraints implying $\lambda_\tau(\tau, \vec{\sigma}) \approx 0$ and the following residual gauge freedom

$$\begin{aligned} A_\tau(\tau, \vec{\sigma}) &\approx A_\tau(\vec{\sigma}), & \Delta A_\tau(\vec{\sigma}) &\approx 0, \\ \eta(\tau, \vec{\sigma}) &\approx \eta_o(\vec{\sigma}) + \tau A_\tau(\vec{\sigma}), \end{aligned} \quad (3.13)$$

Therefore the gauge function $A_\tau(\vec{\sigma})$ must be harmonic. But, since the electro-magnetic potential is assumed to vanish at spatial infinity, this means $A_\tau(\vec{\sigma}) = 0$ and $\eta(\tau, \vec{\sigma}) \approx \eta_o(\vec{\sigma})$. As a consequence these two Killing constraints imply: i) the gauge fixing constraint $A_\tau(\tau, \vec{\sigma}) \approx 0$ for $\pi^\tau(\tau, \vec{\sigma}) \approx 0$; ii) the restriction of the gauge fixing constraint for the Gauss law $\Gamma(\tau, \vec{\sigma}) \approx 0$ to the form $\eta(\tau, \vec{\sigma}) - \eta_o(\vec{\sigma}) \approx 0$ with $\eta_o(\vec{\sigma})$ an arbitrary function. This is a family of gauges with only a residual τ -independent longitudinal gauge freedom and we get $\varphi_r(\tau, \vec{\sigma}) = \pi_{\perp r}(\tau, \vec{\sigma}) \stackrel{\circ}{=} -\partial_\tau A_{\perp r}(\tau, \vec{\sigma}) \approx 0$, namely $A_{\perp r}(\tau, \vec{\sigma}) = A_{\perp r}(\vec{\sigma})$.

The remaining four Killing constraints correspond to the transverse parts of $\varphi_r(\tau, \vec{\sigma}) \approx 0$ and $\psi_r(\tau, \vec{\sigma}) \approx 0$, i.e. they are $\varphi_{\perp r}(\tau, \vec{\sigma}) = \pi_{\perp r}(\tau, \vec{\sigma}) \approx 0$ and $\psi_{\perp r}(\tau, \vec{\sigma}) = \Delta A_{\perp r}(\vec{\sigma}) \approx 0$. Again the boundary conditions at spatial infinity imply that the harmonic functions $A_{\perp r}(\vec{\sigma})$

vanish. Therefore these Killing constraints form the two pairs of second class constraints $\pi_{\perp r}(\tau, \vec{\sigma}) \approx 0$, $A_{\perp r}(\tau, \vec{\sigma}) \approx 0$ killing the DO's of the electro-magnetic field, i.e. its transverse radiative components. Finally the consistency conditions $\partial_\tau \psi_A(\tau, \vec{\sigma}) \approx 0$ are identically satisfied.

In conclusion, with the Killing vector field $X = \partial_\tau$ the allowed set of electro-magnetic configurations is composed only by *pure gauge configurations*. This leaves space only for the introduction of static magnetic monopoles. The addition of another Killing vector field will only reduce the residual gauge freedom of this pure longitudinal gauge configuration.

One can study the effect of the imposition of the other nine Killing symmetries on the electro-magnetic field in the same way. While a Killing symmetry associated with a spatial translation in direction "i" can be shown to imply that the electro-magnetic field does not depend on σ^i , a rotational Killing symmetry implies electro-magnetic fields rotationally invariant around an axis. Finally it can be shown that the Killing symmetry under a boost in direction "i" implies that there is only a static magnetic field produced by a potential $A_{\perp a}(\sigma^{b \neq i})$ and no electric field: therefore there is no genuine radiation field.

IV. TETRAD GRAVITY IN ASYMPTOTICALLY MINKOWSKIAN SPACE-TIMES

After a review of canonical ADM tetrad gravity [16–20] we will study the Hamiltonian constraints implied by a Killing symmetry.

A. The York Canonical Basis for ADM Tetrad Gravity

In ADM tetrad gravity [16–20] the 4-scalar 4-metric defined in Section II is decomposed on cotetrads ${}^4g_{AB}(\tau, \vec{\sigma}) = E_A^{(\alpha)}(\tau, \vec{\sigma}) {}^4\eta_{(\alpha)(\beta)} E_B^{(\beta)}(\tau, \vec{\sigma})$; they are the dynamical variables. The associated tetrads ${}^4E_{(\alpha)}^A(\tau, \vec{\sigma})$ ((α) are flat indices) are connected with the world tetrads ${}^4E_{(\alpha)}^\mu(\tau, \vec{\sigma}) = z_A^\mu(\tau, \vec{\sigma}) {}^4E_{(\alpha)}^A(\tau, \vec{\sigma})$ by using the embedding $z^\mu(\tau, \vec{\sigma})$ of the instantaneous 3-spaces. As said in Section II we have $z_\tau^\mu = (1 + n) l^\mu + N^r z_r^\mu$ with $N^r = n_{(a)} {}^3e_{(a)}^r$, where ${}^r e_{(a)}^r$ are triads on the 3-space Σ_τ . The tetrads admit the following decomposition

$${}^4E_{(\alpha)}^A = {}^4\overset{\circ}{E}_{(o)}^{\overset{\circ}{A}} L^{(o)}_{(\alpha)}(\varphi_{(c)}) + {}^4\overset{\circ}{E}_{(b)}^{\overset{\circ}{A}} R_{(b)(a)}^T(\alpha_{(c)}) L^{(a)}_{(\alpha)}(\varphi_{(c)}), \quad (4.1)$$

where $\varphi_{(a)}(\tau, \vec{\sigma})$ and $\alpha_{(a)}(\tau, \vec{\sigma})$ are the boost and rotation variables of the $O(3,1)$ gauge

freedom in the choice of the tetrads and of their transport. The following barred variables are independent from the angles $\alpha_{(a)}$: $\bar{n}_{(a)} = \sum_b n_{(b)} R_{(b)(a)}(\alpha_{(e)})$, ${}^3e_{(a)}^r = R_{(a)(b)}(\alpha_{(e)}) {}^3\bar{e}_{(b)}^r$.

In Eqs.(4.1) there are the following tetrads and cotetrads adapted to the chosen 3-space Σ_τ ($l_A = z_A^\mu l_\mu$)

$$\begin{aligned} {}^4\bar{\bar{E}}_{(o)}^A &= \frac{1}{1+n} (1; -\bar{n}_{(a)} {}^3\bar{e}_{(a)}^r) = l^A, & {}^4\bar{\bar{E}}_{(a)}^A &= (0; {}^3\bar{e}_{(a)}^r), \\ {}^4\bar{\bar{E}}_A^{(o)} &= (1+n) (1; \vec{0}) = \epsilon l_A, & {}^4\bar{\bar{E}}_A^{(a)} &= (\bar{n}_{(a)}; {}^3\bar{e}_{(a)r}). \end{aligned} \quad (4.2)$$

As shown in Refs. [16, 19] the natural configuration variables of ADM tetrad gravity are $\varphi_{(a)}$, n , $n_{(a)}$, ${}^3e_{(a)r}$. The conjugate momenta are $\pi_{\varphi_{(a)}}$, π_n , $\pi_{n_{(a)}}$, ${}^3\pi_{(a)}^r$. There are 14 (ten primary and four secondary) first-class constraints: seven of the primary ones are $\pi_{\varphi_{(a)}} \approx 0$, $\pi_n \approx 0$, $\pi_{n_{(a)}} \approx 0$.

In Ref.[19] a canonical basis adapted to all the 10 primary first-class constraints was found with a Shanmugadhasan canonical transformation. It leads to the following York canonical basis (see Ref.[19] for the boundary conditions at spatial infinity of the canonical variables)

$$\begin{array}{|c|c|c|c|} \hline \varphi_{(a)} & n & n_{(a)} & {}^3e_{(a)r} \\ \hline \pi_{\varphi_{(a)}} \approx 0 & \pi_n \approx 0 & \pi_{n_{(a)}} \approx 0 & {}^3\pi_{(a)}^r \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline \varphi_{(a)} & \alpha_{(a)} & n & \bar{n}_{(a)} & \theta^r & \tilde{\phi} & R_{\bar{a}} \\ \hline \pi_{\varphi_{(a)}} \approx 0 & \pi_{\alpha_{(a)}} \approx 0 & \pi_n \approx 0 & \pi_{\bar{n}_{(a)}} \approx 0 & \pi_r^{(\theta)} & \pi_{\tilde{\phi}} & \Pi_{\bar{a}} \\ \hline \end{array} \quad (4.3)$$

The secondary first-class constraints are the super-Hamiltonian and super-momentum ones: they are partial differential equations for the determination of $\tilde{\phi}$ and $\pi_r^{(\theta)}$ in terms of θ^r , $\pi_{\tilde{\phi}}$, $R_{\bar{a}}$, $\Pi_{\bar{a}}$.

Due to the use of radar 4-coordinates all the canonical variables of the York basis are 4-scalars of the space-time, but they have different 3-tensorial behaviors inside the 3-spaces. θ^i and $\pi_{\tilde{\phi}}$ are the primary inertial gauge variables, while n and $\bar{n}_{(a)}$ are the secondary ones.

In the York canonical basis we have (from now on we will use V_{ra} for $V_{ra}(\theta^n)$ to simplify

the notation)⁵

$$\begin{aligned}
Q_a &= e^{\sum_{\bar{a}}^{1,2} \gamma_{\bar{a}a} R_{\bar{a}}}, & \tilde{\phi} &= \sqrt{\det {}^3g}, & \pi_{\tilde{\phi}} &= \frac{c^3}{12\pi G} {}^3K, \\
{}^3e_{(a)r} &= \sum_b R_{(a)(b)}(\alpha_{(e)}) {}^3\bar{e}_{(b)r}, & {}^3\bar{e}_{(a)r} &= \tilde{\phi}^{1/3} Q_a V_{ra}, \\
{}^3e_{(a)}^r &= \sum_b R_{(a)(b)}(\alpha_{(e)}) {}^3\bar{e}_{(b)}^r, & {}^3\bar{e}_{(a)}^r &= \tilde{\phi}^{-1/3} Q_a^{-1} V_{ra}, \\
{}^4g_{\tau\tau} &= \epsilon \left[(1+n)^2 - \sum_a \bar{n}_{(a)}^2 \right], \\
{}^4g_{\tau r} &= -\epsilon \sum_a \bar{n}_{(a)} {}^3\bar{e}_{(a)r} = -\epsilon \tilde{\phi}^{1/3} \sum_a Q_a V_{ra} \bar{n}_{(a)}, \\
{}^4g_{rs} &= -\epsilon {}^3g_{rs} = -\epsilon \sum_a {}^3\bar{e}_{(a)r} {}^3\bar{e}_{(a)s} = -\epsilon \tilde{\phi}^{2/3} \sum_a Q_a^2 V_{ra} V_{sa}, \\
{}^3g^{rs} &= \tilde{\phi}^{-2/3} \sum_a Q_a^{-2} V_{ra} V_{sa}, \\
{}^4g^{\tau\tau} &= \frac{\epsilon}{(1+n)^2}, & {}^4g^{\tau r} &= -\epsilon \tilde{\phi}^{-1/3} \frac{Q_a^{-1} V_{ra} \bar{n}_{(a)}}{(1+n)^2}, \\
{}^4g^{rs} &= -\epsilon \tilde{\phi}^{-2/3} Q_a^{-1} Q_b^{-1} V_{ra} V_{sb} \left(\delta_{(a)(b)} - \frac{\bar{n}_{(a)} \bar{n}_{(b)}}{(1+n)^2} \right). \tag{4.4}
\end{aligned}$$

$\alpha_{(a)}(\tau, \vec{\sigma})$ and $\varphi_{(a)}(\tau, \vec{\sigma})$ are the 6 configuration variables parametrizing the $O(3,1)$ gauge freedom in the choice of the tetrads in the tangent plane to each point of Σ_τ and describe the arbitrariness in the choice of a tetrad to be associated to a time-like observer, whose world-line goes through the point $(\tau, \vec{\sigma})$. They fix *the unit 4-velocity of the observer and the conventions for the orientation of gyroscopes and their transport along the world-line of the observer*. The gauge variables $\theta^i(\tau, \vec{\sigma})$, $n(\tau, \vec{\sigma})$, $\bar{n}_{(a)}(\tau, \vec{\sigma})$ describe inertial effects, which are the relativistic counterpart of the non-relativistic ones (the centrifugal, Coriolis,... forces in Newton mechanics in accelerated frames) and which are present also in the non-inertial frames of Minkowski space-time [25].

In Eq.(4.4) the quantity ${}^3K(\tau, \vec{\sigma})$ is the trace of the extrinsic curvature ${}^3K_{rs}(\tau, \vec{\sigma})$ of the instantaneous 3-spaces Σ_τ whose expression is given in Appendix A. This so-called

⁵ The set of numerical parameters $\gamma_{\bar{a}a}$ satisfies $\sum_u \gamma_{\bar{a}u} = 0$, $\sum_u \gamma_{\bar{a}u} \gamma_{bu} = \delta_{\bar{a}b}$, $\sum_{\bar{a}} \gamma_{\bar{a}u} \gamma_{\bar{a}v} = \delta_{uv} - \frac{1}{3}$. Each solution of these equations defines a different York canonical basis.

York time ${}^3K(\tau, \vec{\sigma})$ is the only gauge variable among the momenta: this is a reflex of the Lorentz signature of space-time, because $\pi_{\tilde{\phi}}(\tau, \vec{\sigma})$ and $\theta^n(\tau, \vec{\sigma})$ can be used as a set of 4-coordinates [19, 20, 31]. Its conjugate variable, to be determined by the super-Hamiltonian constraint, is the conformal factor of the 3-metric $\tilde{\phi}(\tau, \vec{\sigma})$, which is the *3-volume density* on Σ_τ : $V_R = \int_R d^3\sigma \tilde{\phi}(\tau, \vec{\sigma})$, $R \subset \Sigma_\tau$.

The two pairs of canonical variables $R_{\bar{a}}(\tau, \vec{\sigma})$, $\Pi_{\bar{a}}(\tau, \vec{\sigma})$, $\bar{a} = 1, 2$, describe the generalized *tidal effects*, namely the independent degrees of freedom of the gravitational field. In particular the configuration tidal variables $R_{\bar{a}}$ depend *only on the eigenvalues of the 3-metric*. They are DO *only* with respect to the gauge transformations generated by 10 of the 14 first class constraints. Let us remark that, if we fix completely the gauge and we go to Dirac brackets, then the only surviving dynamical variables $R_{\bar{a}}$ and $\Pi_{\bar{a}}$ become two pairs of *non canonical* DO for that gauge (see Ref.[21] for new results on the DO's).

The Dirac Hamiltonian is (the λ 's are arbitrary Dirac multipliers)

$$\begin{aligned} H_D = & E_{ADM} + \int d^3\sigma \left[-\epsilon c n \mathcal{H} + \bar{n}_{(a)} \bar{\mathcal{H}}_{(a)} \right] (\tau, \vec{\sigma}) + \\ & + \int d^3\sigma \left[\lambda_n \pi_n + \lambda_{\bar{n}_{(a)}} \pi_{\bar{n}_{(a)}} + \lambda_{\varphi_{(a)}} \pi_{\varphi_{(a)}} + \lambda_{\alpha_{(a)}} \pi_{\alpha_{(a)}} \right] (\tau, \vec{\sigma}). \end{aligned} \quad (4.5)$$

See Eqs. (3.45), (B.8), (3.42) and (3.44) of the first paper in Ref.[20] for the expression of the ADM energy E_{ADM} and of the super-Hamiltonian, \mathcal{H} , and super-momentum, $\bar{\mathcal{H}}_{(a)}$, constraints: all these quantities are functions of θ^r , $\pi_r^{(\theta)}$, $\tilde{\phi}$, $\pi_{\tilde{\phi}}$, $R_{\bar{a}}$, $\Pi_{\bar{a}}$, but not of n and $\bar{n}_{(a)}$.

In the following we shall work in the *Schwinger time gauge* $\varphi_{(a)}(\tau, \vec{\sigma}) \approx 0$ (tetrads adapted to the 3+1 splitting) and $\alpha_{(a)}(\tau, \vec{\sigma}) \approx 0$ (arbitrary choice of an origin for rotations), where ${}^3e_{(a)r}(\tau, \vec{\sigma}) \approx {}^3\bar{e}_{(a)r}(\tau, \vec{\sigma})$, $\lambda_{\varphi_{(a)}}(\tau, \vec{\sigma}) = \lambda_{\alpha_{(a)}}(\tau, \vec{\sigma}) = 0$.

The first kinematical half of Hamilton equations implies the following expressions for the ∂_τ derivatives (the velocities)

$$\begin{aligned}
\partial_\tau n(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \lambda_n(\tau, \vec{\sigma}), \\
\partial_\tau \bar{n}_{(a)}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \lambda_{\bar{n}_{(a)}}(\tau, \vec{\sigma}), \\
\partial_\tau {}^3\bar{e}_{(a)r}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \left[- (1+n) {}^3K_{rs} {}^3\bar{e}_{(a)}^s + \partial_r \bar{n}_{(a)} + \bar{n}_{(b)} {}^3\bar{e}_{(b)}^s (\partial_s {}^3\bar{e}_{(a)r} - \partial_r {}^3\bar{e}_{(a)s}) \right] (\tau, \vec{\sigma}), \\
\\
\partial_\tau {}^4g_{\tau\tau}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} 2\epsilon \left[(1+n) \lambda_n - \bar{n}_{(a)} \lambda_{\bar{n}_{(a)}} \right] (\tau, \vec{\sigma}), \\
\partial_\tau {}^4g_{\tau r}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} -\epsilon \left[\lambda_{\bar{n}_{(a)}} {}^3\bar{e}_{(a)r} + \bar{n}_{(a)} \left(- (1+n) {}^3K_{rs} {}^3\bar{e}_{(a)}^s + \right. \right. \\
&\quad \left. \left. + \partial_r \bar{n}_{(a)} + \bar{n}_{(b)} {}^3\bar{e}_{(b)}^s (\partial_s {}^3\bar{e}_{(a)r} - \partial_r {}^3\bar{e}_{(a)s}) \right) \right] (\tau, \vec{\sigma}), \\
\partial_\tau {}^4g_{rs}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \left[\partial_r (\bar{n}_{(a)} {}^3\bar{e}_{(a)s}) + \partial_s (\bar{n}_{(a)} {}^3\bar{e}_{(a)r}) - 2 {}^3\Gamma_{rs}^u \bar{n}_{(a)} {}^3\bar{e}_{(a)u} - 2 (1+n) {}^3K_{rs} \right] (\tau, \vec{\sigma}).
\end{aligned} \tag{4.6}$$

B. The Killing Equations Associated to the Given Killing Vector Field X .

We shall assume that the 4-metric is left invariant by a given Killing vector field $X = \xi^A(\tau, \vec{\sigma}) \partial_A$: $L_X {}^4g_{AB}(\tau, \vec{\sigma}) d\sigma^A d\sigma^B = 0$. With generic tetrads one has $L_X {}^4E_A^{(\alpha)}(\tau, \vec{\sigma}) d\sigma^A \neq 0$. However as shown in Ref. [32] and in its bibliography, the existence of the Killing vector X for the 4-metric implies that there is a special set of tetrads ${}^4\tilde{E}_A^{(\alpha)}(\tau, \vec{\sigma})$ such that $L_X {}^4\tilde{E}_A^{(\alpha)}(\tau, \vec{\sigma}) d\sigma^A = 0$.

The existence of the Killing vector implies the 10 Killing equations

$$\chi_{AB}(\tau, \vec{\sigma}) = \left({}^4\nabla_A \xi_B + {}^4\nabla_B \xi_A \right) (\tau, \vec{\sigma}) = \left(\partial_A \xi_B + \partial_B \xi_A - 2 {}^4\Gamma_{AB}^C \xi_C \right) (\tau, \vec{\sigma}) = 0. \tag{4.7}$$

By using the notation of the previous Subsection we have

$$\xi_A = {}^4g_{AB} \xi^B, \quad \xi^A = {}^4g^{AB} \xi_A,$$

$$\begin{aligned} \xi_\tau &= \epsilon \left[\left((1+n)^2 - \bar{n}_{(a)} \bar{n}_{(a)} \right) \xi^\tau - \bar{n}_{(a)} {}^3\bar{e}_{(a)r} \xi^r \right], \\ \xi_r &= -\epsilon \left[\bar{n}_{(a)} {}^3\bar{e}_{(a)r} \xi^\tau + {}^3g_{rs} \xi^s \right], \\ \xi^\tau &= \frac{\epsilon}{(1+n)^2} \left[\xi_\tau - \bar{n}_{(a)} {}^3\bar{e}_{(a)}^r \xi_r \right], \\ \xi^r &= -\epsilon {}^3\bar{e}_{(a)}^r \left[{}^3\bar{e}_{(a)}^s \xi_s + \frac{\bar{n}_{(a)} (\xi_\tau - \bar{n}_{(b)} {}^3\bar{e}_{(b)}^s \xi_s)}{(1+n)^2} \right]. \end{aligned} \quad (4.8)$$

By using Eqs.(4.6) for the time-derivative of the metric and Eq.(4.4) for its spatial derivatives $(\partial_r {}^4g_{\tau\tau} \stackrel{\circ}{=} 2\epsilon \left[(1+n) \partial_r n - \bar{n}_{(a)} \partial_r \bar{n}_{(a)} \right], \partial_s {}^4g_{\tau r} = -\epsilon \partial_s (\bar{n}_{(a)} {}^3\bar{e}_{(a)r}), \partial_u {}^4g_{rs} = -\epsilon \partial_u {}^3g_{rs} = -\epsilon ({}^3\bar{e}_{(a)r} \partial_u {}^3\bar{e}_{(a)s} + {}^3\bar{e}_{(a)s} \partial_u {}^3\bar{e}_{(a)r})$), we get

$$\begin{aligned} \partial_\tau \xi_\tau &\stackrel{\circ}{=} \epsilon \left[\left((1+n)^2 - \bar{n}_{(a)} \bar{n}_{(a)} \right) \partial_\tau \xi^\tau - \bar{n}_{(a)} {}^3\bar{e}_{(a)r} \partial_\tau \xi^r + \right. \\ &\quad + 2 \left((1+n) \lambda_n - \bar{n}_{(a)} \lambda_{\bar{n}_{(a)}} \right) \xi^\tau - \left(\lambda_{\bar{n}_{(a)}} {}^3\bar{e}_{(a)r} + \bar{n}_{(a)} \left[\partial_r \bar{n}_{(a)} + \right. \right. \\ &\quad \left. \left. + \bar{n}_{(b)} {}^3\bar{e}_{(b)}^s (\partial_s {}^3\bar{e}_{(a)r} - \partial_r {}^3\bar{e}_{(a)s}) - (1+n) {}^3K_{rv} {}^3\bar{e}_{(a)}^v \right] \right) \xi^r \Big], \end{aligned}$$

$$\begin{aligned} \partial_r \xi_\tau &= \epsilon \left[\left((1+n)^2 - \bar{n}_{(a)} \bar{n}_{(a)} \right) \partial_r \xi^\tau - \bar{n}_{(a)} {}^3\bar{e}_{(a)s} \partial_r \xi^s + \right. \\ &\quad \left. + 2 \left((1+n) \partial_r n - \bar{n}_{(a)} \partial_r \bar{n}_{(a)} \right) \xi^\tau - \partial_r (\bar{n}_{(a)} {}^3\bar{e}_{(a)s}) \xi^s \right], \end{aligned} \quad (4.9)$$

$$\begin{aligned} \partial_\tau \xi_r &\stackrel{\circ}{=} -\epsilon \left[\bar{n}_{(a)} {}^3\bar{e}_{(a)r} \partial_\tau \xi^\tau + {}^3g_{rs} \partial_\tau \xi^s + \right. \\ &\quad + \left(\lambda_{\bar{n}_{(a)}} {}^3\bar{e}_{(a)r} + \bar{n}_{(a)} \left[\partial_r \bar{n}_{(a)} + \bar{n}_{(b)} {}^3\bar{e}_{(b)}^s (\partial_s {}^3\bar{e}_{(a)r} - \partial_r {}^3\bar{e}_{(a)s}) - (1+n) {}^3K_{rv} {}^3\bar{e}_{(a)}^v \right] \right) \xi^\tau + \\ &\quad \left. + \left(\bar{n}_{(a)} \left[\partial_r {}^3\bar{e}_{(a)s} + \partial_s {}^3\bar{e}_{(a)r} - 2 {}^3\Gamma_{rs}^u {}^3\bar{e}_{(a)u} \right] + \partial_r \bar{n}_{(a)} {}^3\bar{e}_{(a)s} + \partial_s \bar{n}_{(a)} {}^3\bar{e}_{(a)r} - 2(1+n) {}^3K_{rs} \right) \xi^s \right], \end{aligned}$$

$$\begin{aligned} \partial_r \xi_s &= -\epsilon \left[\bar{n}_{(a)} {}^3\bar{e}_{(a)s} \partial_r \xi^\tau + {}^3g_{sv} \partial_r \xi^v + \partial_r (\bar{n}_{(a)} {}^3\bar{e}_{(a)s}) \xi^\tau + ({}^3\bar{e}_{(a)s} \partial_r {}^3\bar{e}_{(a)v} + {}^3\bar{e}_{(a)v} \partial_r {}^3\bar{e}_{(a)s}) \xi^v \right]. \end{aligned} \quad (4.10)$$

C. The Hamiltonian Expression of the Killing Equations.

By using Eqs.(A1) and (4.8) we get the following expression for the 10 Killing constraints implied by the Killing equations (4.7)

$$\begin{aligned}
\frac{1}{2} \chi_{\tau\tau} &= \partial_\tau \xi_\tau - ({}^4\Gamma_{\tau\tau}^\tau \xi_\tau + {}^4\Gamma_{\tau\tau}^u \xi_u) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} \epsilon \left[\left((1+n)^2 - \bar{n}_{(a)} \bar{n}_{(a)} \right) \partial_\tau \xi^\tau - \bar{n}_{(a)} {}^3\bar{e}_{(a)r} \partial_\tau \xi^r + \right. \\
&+ \left[(1+n) \lambda_n - \bar{n}_{(a)} \lambda_{\bar{n}_{(a)}} \right] \xi^\tau + \\
&+ \left((1+n) {}^3\bar{e}_{(a)r} \left[{}^3\bar{e}_{(a)}^s \partial_s n - {}^3\bar{K}_{(a)(b)} \bar{n}_{(b)} \right] - \right. \\
&\left. \left. - \bar{n}_{(a)} \left[\partial_r \bar{n}_{(a)} + \bar{n}_{(b)} {}^3\bar{e}_{(b)}^s (\partial_s {}^3\bar{e}_{(a)r} - \partial_s {}^3\bar{e}_{(a)s}) \right] \right) \xi^r \right] \approx 0, \tag{4.11}
\end{aligned}$$

$$\begin{aligned}
\chi_{\tau r} &= \partial_\tau \xi_r + \partial_r \xi_\tau - 2 ({}^4\Gamma_{\tau r}^\tau \xi_\tau + {}^4\Gamma_{\tau r}^u \xi_u) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} \epsilon \left[\left((1+n)^2 - \bar{n}_{(a)} \bar{n}_{(a)} \right) \partial_r \xi^\tau - \bar{n}_{(a)} {}^3\bar{e}_{(a)s} \partial_r \xi^s - \right. \\
&- \bar{n}_{(a)} {}^3\bar{e}_{(a)r} \partial_\tau \xi^\tau - {}^3g_{rs} \partial_\tau \xi^s - \\
&- \left({}^3\bar{e}_{(a)r} \lambda_{\bar{n}_{(a)}} + \bar{n}_{(a)} \partial_r \bar{n}_{(a)} + (1+n) {}^3\bar{K}_{r(a)} \bar{n}_{(a)} + \right. \\
&\left. + \bar{n}_{(a)} \bar{n}_{(b)} {}^3\bar{e}_{(b)}^s (\partial_s {}^3\bar{e}_{(a)r} - \partial_r {}^3\bar{e}_{(a)s}) \right) \xi^\tau - \partial_s \left(\bar{n}_{(a)} {}^3\bar{e}_{(a)r} \right) \xi^s \right] \approx 0, \tag{4.12}
\end{aligned}$$

$$\begin{aligned}
\chi_{rs} &= \partial_r \xi_s + \partial_s \xi_r - 2 ({}^4\Gamma_{rs}^\tau \xi_\tau + {}^4\Gamma_{rs}^u \xi_u) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} -\epsilon \left[\bar{n}_{(a)} {}^3\bar{e}_{(a)s} \partial_r \xi^\tau + {}^3g_{sv} \partial_r \xi^v + \bar{n}_{(a)} {}^3\bar{e}_{(a)r} \partial_s \xi^\tau + {}^3g_{rv} \partial_s \xi^v + \right. \\
&+ \left[-2(1+n) {}^3K_{rs} + \partial_r (\bar{n}_{(a)} {}^3\bar{e}_{(a)s}) + \partial_s (\bar{n}_{(a)} {}^3\bar{e}_{(a)r}) - \right. \\
&\left. - \bar{n}_{(a)} {}^3\bar{e}_{(a)}^v (\partial_r {}^3g_{sv} + \partial_s {}^3g_{rv} - \partial_v {}^3g_{rs}) \right] \xi^\tau - \xi^v \partial_v {}^3g_{rs} \right] \approx 0, \tag{4.13}
\end{aligned}$$

By consistency we must have

$$\psi_{AB}(\tau, \vec{\sigma}) = \partial_\tau \chi_{AB}(\tau, \vec{\sigma}) + \{\chi_{AB}(\tau, \vec{\sigma}), H_D\}, \approx 0, \tag{4.14}$$

where we have to use Eqs.(4.5) for the Dirac Hamiltonian with $\lambda_{\varphi_{(a)}}(\tau, \vec{\sigma}) = \lambda_{\alpha_{(a)}}(\tau, \vec{\sigma}) = 0$ (Schwinger time gauges) and where ∂_τ acts on $\xi^A(\tau, \vec{\sigma})$.

to evaluate the velocities.

In the Schwinger time gauges the 16 variables consisting in the 8 gauge variables $n, \bar{n}_{(a)}, \theta^r, \pi_{\tilde{\phi}},$ in the 4 physical tidal variables $R_{\bar{a}}, \Pi_{\bar{a}},$ and in the 4 Dirac multipliers $\lambda_n, \lambda_{\bar{n}_{(a)}} (with $\tilde{\phi}$ and $\pi_r^{(\theta)}$ determined by the secondary first-class constraints $\mathcal{H}(\tau, \vec{\sigma}) \approx 0, \bar{\mathcal{H}}_{(a)}(\tau, \vec{\sigma}) \approx 0$ as functions of $\theta^r, \pi_{\tilde{\phi}}, R_{\bar{a}}, \Pi_{\bar{a}}$) are restricted by the 20 Killing constraints $\chi_{AB}(\tau, \vec{\sigma}) \approx 0, \psi_{AB}(\tau, \vec{\sigma}) \approx 0$. Therefore some of these Killing constraints must be void not to have over-determined equations and moreover this implies that we must have $\partial_\tau \psi_{AB}(\tau, \vec{\sigma}) = 0$ automatically satisfied.$

D. A Time-like Killing Vector

Let us try to understand the meaning of these Killing constraints for $X = \partial_\tau$ (*stationary* space-times). We have $\xi^\tau = 1$ and $\xi^r = 0$, so that $\xi_\tau = \epsilon \left((1+n)^2 - \bar{n}_{(a)} \bar{n}_{(a)} \right), \xi_r = -\epsilon \bar{n}_{(a)} {}^3\bar{e}_{(a)r}.$

$$\chi_{\tau\tau} \stackrel{\circ}{=} 2\epsilon \left((1+n) \lambda_n - n_{(a)} \lambda_{\bar{n}_{(a)}} \right) \approx 0, \quad (4.15)$$

$$\begin{aligned} \chi_{\tau r} \stackrel{\circ}{=} & -\epsilon \left({}^3\bar{e}_{(a)r} \lambda_{\bar{n}_{(a)}} + \bar{n}_{(a)} \partial_r \bar{n}_{(a)} + \right. \\ & \left. + \bar{n}_{(a)} \bar{n}_{(b)} {}^3\bar{e}_{(b)}^v (\partial_v {}^3\bar{e}_{(a)r} - \partial_r {}^3\bar{e}_{(a)v}) - (1+n) {}^3K_{rv} {}^3\bar{e}_{(a)}^v \right) \approx 0, \end{aligned} \quad (4.16)$$

$$\begin{aligned} \chi_{rs} = & -\epsilon \left(-2(1+n) {}^3K_{rs} + \partial_r (\bar{n}_{(a)} {}^3\bar{e}_{(a)s}) + \partial_s (\bar{n}_{(a)} {}^3\bar{e}_{(a)r}) - \right. \\ & \left. - \bar{n}_{(a)} {}^3\bar{e}_{(a)}^v (\partial_r {}^3g_{sv} + \partial_s {}^3g_{rv} - \partial_v {}^3g_{rs}) \right) \approx 0. \end{aligned} \quad (4.17)$$

As a consequence we get $({}^3\bar{K}_{(a)(b)} = {}^3\bar{e}_{(a)}^r {}^3\bar{e}_{(b)}^s {}^3K_{rs}, {}^3\bar{K}_{r(a)} = {}^3\bar{e}_{(a)}^s {}^3K_{rs})$

$$\begin{aligned} \lambda_n \stackrel{\circ}{=} & \partial_\tau n \approx \frac{\bar{n}_{(a)} \lambda_{\bar{n}_{(a)}}}{1+n}, \\ \lambda_{\bar{n}_{(a)}} \stackrel{\circ}{=} & \partial_\tau \bar{n}_{(a)} \approx -{}^3\bar{e}_{(a)}^r \left[\bar{n}_{(b)} \partial_r \bar{n}_{(b)} + (1+n) {}^3\bar{K}_{r(b)} \bar{n}_{(b)} + \bar{n}_{(b)} \bar{n}_{(c)} {}^3\bar{e}_{(c)}^v (\partial_v {}^3\bar{e}_{(b)r} - \partial_r {}^3\bar{e}_{(b)v}) \right], \\ {}^3K_{rs} \approx & \frac{1}{2(1+n)} \left[\partial_r (\bar{n}_{(a)} {}^3\bar{e}_{(a)s}) + \partial_s (\bar{n}_{(a)} {}^3\bar{e}_{(a)r}) - \right. \\ & \left. - \bar{n}_{(a)} {}^3\bar{e}_{(a)}^v (\partial_r {}^3g_{sv} + \partial_s {}^3g_{rv} - \partial_v {}^3g_{rs}) \right], \end{aligned} \quad (4.18)$$

with ${}^3g_{rs} = \tilde{\phi}^2/3 \sum_a Q_a^2 V_{ra}(\theta^u) V_{sa}(\theta^u)$ from Eq.(4.4) and with ${}^3K_{rs}$ given in Eq.(A1) as a function of the canonical variables in the York canonical basis.

As in the electro-magnetic case, the Killing constraints $\chi_{\tau A} \approx 0$ restrict the gauge freedom by determining the 4 Dirac multipliers λ_n and $\lambda_{\bar{n}_{(a)}}$ associated to lapse and shift.

The extra constraints $\psi_{\tau A} \approx 0$ will be automatically satisfied (like in the electro-magnetic case) since they determine the velocities $\partial_\tau \lambda_n$ and $\partial_\tau \lambda_{\bar{n}_{(a)}}$ of the already determined Dirac multipliers.

The traces of the constraints $\chi_{rs} \approx 0$ and $\psi_{rs} \approx 0$ determine the gauge variable 3K ($\pi_{\tilde{\phi}}$), namely the clock synchronization convention, and the lapse function n , namely they determine the primary and secondary gauge variables associated with the Dirac multiplier λ_n . Like in the electro-magnetic case some residual τ -independent gauge freedom can be left by the chosen boundary conditions at spatial infinity for the tetrads.

The 10 constraints from the traceless part of the constraints $\chi_{rs} \approx 0$ and $\psi_{rs} \approx 0$ are the equations for the determination of the 3 primary θ^r and 3 secondary $\bar{n}_{(a)}$ gauge variables associated with the Dirac multiplier $\lambda_{\bar{n}_{(a)}}$ and of the four DO's $R_{\bar{a}}, \Pi_{\bar{a}}$.

Therefore no physical tidal variables (no gravitational waves in the linearized theory) survive to the presence of a time-like Killing symmetry. Only a τ -independent gauge freedom is left and only static singularities like black holes are allowed.

With non-time-like Killing symmetries tidal variables adapted to the symmetry would survive.

V. CONCLUSIONS

In this paper the lacking Hamiltonian formulation of Killing symmetries in terms of Dirac constraints added by hand was given. This was done both for the electro-magnetic field and for tetrad gravity in asymptotically Minkowskian space-times.

It was shown that in both cases the presence of a time-like Killing symmetry kills all the physical degrees of freedom, namely the DO's. This result was known to many people but is not present in the literature as far as we know.

Appendix A: Hamiltonian Expressions

The Hamiltonian expression [20, 21] of the extrinsic curvature ${}^3K_{rs}$ of the instantaneous 3-spaces Σ_τ and of the 4-Christoffel symbols ${}^4\Gamma_{BC}^A = \frac{1}{2} {}^4g^{AE} \left(\partial_B {}^4g_{CE} + \partial_C {}^4g_{BE} - \partial_E {}^4g_{BC} \right)$ is

$$\begin{aligned} {}^3\tilde{K}_{rs} = & \epsilon \frac{4\pi G}{c^3} \tilde{\phi}^{-1/3} \left(\sum_a Q_a^2 V_{ra}(\theta^n) V_{sa}(\theta^n) [2 \sum_{\bar{b}} \gamma_{\bar{b}a} \Pi_{\bar{b}} - \tilde{\phi} \pi_{\tilde{\phi}}] + \right. \\ & \left. + \sum_{ab} Q_a Q_b [V_{ra}(\theta^n) V_{sb}(\theta^n) + V_{rb}(\theta^n) V_{sa}(\theta^n)] \sum_{twi} \frac{\epsilon_{abt} V_{tw}(\theta^n) B_{iw}(\theta^n) \pi_i^{(\theta)}}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \right), \end{aligned}$$

$$\begin{aligned} {}^4\Gamma_{\tau\tau}^\tau &= \frac{1}{1+n} \left(\partial_\tau n + \bar{n}_{(a)} {}^3\bar{e}_{(a)}^r \partial_r n - \bar{n}_{(a)} \bar{n}_{(b)} {}^3\bar{K}_{(a)(b)} \right) \stackrel{\circ}{=} \\ &\stackrel{\circ}{=} \frac{1}{1+n} \left(\lambda_n + \bar{n}_{(a)} {}^3\bar{e}_{(a)}^r \partial_r n - \bar{n}_{(a)} \bar{n}_{(b)} {}^3\bar{K}_{(a)(b)} \right), \end{aligned}$$

$${}^4\Gamma_{\tau r}^\tau = \frac{1}{1+n} \left(\partial_r n - {}^3\bar{K}_{r(a)} \bar{n}_{(a)} \right),$$

$${}^4\Gamma_{rs}^\tau = -\frac{1}{1+n} {}^3K_{rs},$$

$$\begin{aligned} {}^4\Gamma_{\tau\tau}^u &\stackrel{\circ}{=} {}^3\bar{e}_{(a)}^u \left[\partial_\tau \bar{n}_{(a)} - \frac{\bar{n}_{(a)}}{1+n} \partial_\tau n + \right. \\ &+ (1+n) \left(\delta_{(a)(b)} - \frac{\bar{n}_{(a)} \bar{n}_{(b)}}{(1+n)^2} \right) \left({}^3\bar{e}_{(b)}^r \partial_r n - {}^3\bar{K}_{(b)(c)} \bar{n}_{(c)} \right) \Big] \stackrel{\circ}{=} \\ &\stackrel{\circ}{=} {}^3\bar{e}_{(a)}^u \left[\bar{\lambda}_{\bar{n}_{(a)}} - \frac{\bar{n}_{(a)}}{1+n} \lambda_n + \right. \\ &+ (1+n) \left(\delta_{(a)(b)} - \frac{\bar{n}_{(a)} \bar{n}_{(b)}}{(1+n)^2} \right) \left({}^3\bar{e}_{(b)}^r \partial_r n - {}^3\bar{K}_{(b)(c)} \bar{n}_{(c)} \right) \Big], \end{aligned}$$

$$\begin{aligned}
{}^4\Gamma_{\tau r}^u &= {}^3\bar{e}_{(a)}^u \left[\partial_r \bar{n}_{(a)} - \frac{\bar{n}_{(a)}}{1+n} \partial_r n + {}^3\bar{\omega}_{r(a)(b)} \bar{n}_{(b)} - \right. \\
&\quad \left. - (1+n) \left(\delta_{(a)(b)} - \frac{\bar{n}_{(a)} \bar{n}_{(b)}}{(1+n)^2} \right) {}^3\bar{K}_{r(b)} \right] = \\
&= {}^3\bar{e}_{(a)}^u \left[\partial_r \bar{n}_{(a)} - \frac{\bar{n}_{(a)}}{1+n} \partial_r n - (1+n) \left(\delta_{(a)(b)} - \frac{\bar{n}_{(a)} \bar{n}_{(b)}}{(1+n)^2} \right) {}^3\bar{K}_{r(b)} \right] + \\
&\quad + \bar{n}_{(a)} \left(\partial_r {}^3\bar{e}_{(a)}^u + {}^3\Gamma_{rs}^u {}^3\bar{e}_{(a)}^s \right), \\
\\
{}^4\Gamma_{rs}^u &= {}^3\Gamma_{rs}^u + \frac{\bar{n}_{(a)}}{1+n} {}^3\bar{e}_{(a)}^u {}^3K_{rs}, \\
{}^3\Gamma_{rs}^u &= \frac{1}{2} {}^3g^{uv} \left(\partial_r {}^3g_{sv} + \partial_s {}^3g_{rv} - \partial_v {}^3g_{rs} \right). \tag{A1}
\end{aligned}$$

-
- [1] A.E.Fischer, *The Theory of Superspace*, in *Relativity*, eds. M.Carmeli, L.Fickler and L.Witten (Plenum, New York, 1970); *Resolving the Singularities in the Space of Riemannian Geometries*, J.Math.Phys. **27**, 718 (1986).
 - [2] J.M.Arms, J.E.Marsden and V.Moncrief, *Symmetries and Bifurcations of Momentum Mappings*, Commun.Math.Phys. **78**, 455 (1981).
 - [3] S.Timothy Swift, *Natural Bundles I. A Minimal Resolution of Superspace*, J.Math.Phys. **33**, 3723 (1992); *Natural Bundles II. Spin and the Diffeomorphism Group*, J.Math.Phys. **34**, 3825 (1993); *Natural Bundles III. Resolving the Singularities in the Space of Immersed Submanifolds*, J.Math.Phys. **34**, 3841 (1993).
 - [4] V.Moncrief, *Decompositions of Gravitational Perturbations*, J.Math.Phys. **16**, 1556 (1975). *Spacetime Symmetries and Linearization Stability of Einstein Equations I and II*, J.Math.Phys. **16**, 493 (1975) and **17**, 1893 (1976).
 - [5] J.Arms, *Symmetries and Solution Set Singularities in Hamiltonian Field Theories*, Acta Phys.Pol. **B17**, 499 (1986). *The Structure of the Solution Set for the Yang-Mills Equations*, Math.Proc.Camb.Phil.Soc. **90**, 361 (1981). *Linearization Stability of Gravitational and Gauge Fields*, J.Math.Phys. **20**, 443 (1979).
 - [6] V.Moncrief, *Gribov Degeneracies: Coulomb Gauge Conditions and Initial Value Constraints*, J.Math.Phys. **20**, 579 (1979).
 - [7] L.Lusanna, *Classical Yang-Mills Theory with Fermions. I. General Properties of a System with Constraints*, Int.J.Mod.Phys. **A10**, 3531-3579 (1995); *II. Dirac's Observables*, Int.J.Mod.Phys. **A10**, 3675-3757 (1995).
 - [8] Y.Choquet-Bruhat, *Positive-Energy Theorems*, in *Relativity, Groups and Topology II*, eds. B.S.De Witt and R.Stora (North-Holland, Amsterdam, 1984).
 - [9] Y.Choquet-Bruhat and J.W.York jr, *The Cauchy Problem*, in *General Relativity and Gravitation*, Vol. 1, pp. 99-172, ed. A.Held (Plenum Press, New York, 1980).
 - [10] M.Cantor, *Elliptic Operators and the Decomposition of Tensor Fields*, Bull.Am.Math.Soc. **5**, 235 (1981); *Some Problems of Global Analysis on Asymptotically Simple Manifolds*, Comp. Math. **38**, 3 (1979); *The Existence of Non-Trivial Asymptotically Flat Initial Data for Vacuum Spacetimes*, Commun.Math.Phys. **57**, 83 (1977).
 - [11] R.Arnowitt, S.Deser and C.W.Misner, *The Dynamics of General Relativity*, ch.7 of *Gravitation: an Introduction to Current Research*, ed. L.Witten (Wiley, New York, 1962)(arXiv gr-qc/0405109).
 - [12] R.Beig and Ó Murchadha, *The Poincaré Group as the Symmetry Group of Canonical General Relativity*, Ann.Phys.(N.Y.) **174**, 463 (1987).
 - [13] P.A.M.Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Monographs Series (Yeshiva University, New York, N.Y., 1964).
 - [14] M.Henneaux and C.Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).
 - [15] L.Lusanna, *The Rest-Frame Instant Form of Metric Gravity*, Gen.Rel.Grav. **33**, 1579-1696 (2001) (gr-qc/0101048).
 - [16] L.Lusanna and S.Russo, *A New Parametrization for Tetrad Gravity*, Gen.Rel.Grav. **34**, 189-242 (2002) (gr-qc/0102074).
 - [17] R.De Pietri, L.Lusanna, L.Martucci and S.Russo, *Dirac's Observables for the Rest-Frame*

- Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge*, Gen.Rel.Grav. **34**, 877-1033 (2002) (gr-qc/0105084).
- [18] J.Agresti, R.De Pietri, L.Lusanna and L.Martucci, *Hamiltonian Linearization of the Rest-Frame Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge: a Radiation Gauge for Background-Independent Gravitational Waves in a Post-Minkowskian Einstein Space-Time*, Gen.Rel.Grav. **36**, 1055-1134 (2004) (gr-qc/0302084).
 - [19] D.Alba and L.Lusanna, *The York Map as a Shanmugadhasan Canonical Transformation in Tetrad Gravity and the Role of Non-Inertial Frames in the Geometrical View of the Gravitational Field*, Gen.Rel.Grav. **39**, 2149 (2007) (gr-qc/0604086, v2).
 - [20] D.Alba and L.Lusanna, *The Einstein-Maxwell-Particle System in the York Canonical Basis of ADM Tetrad Gravity: I) The Equations of Motion in Arbitrary Schwinger Time Gauges.*, (arXiv 0907.4087) Canad.J.Phys. **90**, 1017 (2012); *II) The Weak Field Approximation in the 3-Orthogonal Gauges and Hamiltonian Post-Minkowskian Gravity: the N-Body Problem and Gravitational Waves with Asymptotic Background.*, (arXiv 1003.5143) Canad.J.Phys. **90**, 1077 (2012); *III) The Post-Minkowskian N-Body Problem, its Post-Newtonian Limit in Non-Harmonic 3-Orthogonal Gauges and Dark Matter as an Inertial Effect*, (arXiv 1009.1794) Canad.J.Phys. **90**, 1131 (2012).
 - [21] L.Lusanna and M.Villani, *Hamiltonian Expression of Curvature Tensors in the York Canonical Basis: I) The Riemann Tensor and Ricci Scalars; II) The Weyl Tensor, Weyl Scalars, The Weyl Eigenvalues and the Problem of the Observables of the Gravitational Field*, Int.J.Geom.Meth. Phys. **11**, 1450052 and 1450053 (2014) (arXiv 1401.1370 and 1401.1375).
 - [22] L.Lusanna, *From Clock Synchronization to Dark Matter as a Relativistic Inertial Effect*, Lecture at the *Black Objects in Supergravity School BOSS2011*, Frascati, 9-13 May 2011 (arXiv 1205.2481), Springer Proc.Phys. **144**, pp.267-343 (Springer, Berlin, 2013); *Non-Inertial Frames in Special and General Relativity*, to appear in the Proceedings of the School *Gravity: where do we stand?*, 11-15 May 2009, Villa Olmo, Como, Italy, ed.R.Peron(arXiv:1310.4465); *Canonical ADM Tetrad Gravity: from Metrological Inertial Gauge Variables to Dynamical Tidal Dirac Observables*, Int.J.Geom.Meth.Mod.Phys. **12**, 1530001 (2015) (arXiv 1401.1375).
 - [23] Bondi H., *Assumption and Myth in Physical Theory* (Cambridge University Press, Cambridge, 1967).
 - [24] D'Inverno R., *Introducing Einstein Relativity* (Oxford University Press, Oxford, 1992).
 - [25] D.Alba and L.Lusanna, *Charged Particles and the Electro-Magnetic Field in Non-Inertial Frames: I. Admissible 3+1 Splittings of Minkowski Spacetime and the Non-Inertial Rest Frames*, Int.J.Geom.Methods in Physics **7**, 33 (2010) (arXiv 0908.0213) and *II. Applications: Rotating Frames, Sagnac Effect, Faraday Rotation, Wrap-up Effect* (arXiv 0908.0215), Int.J.Geom.Methods in Physics, **7**, 185 (2010); *Generalized Radar 4-Coordinates and Equal-Time Cauchy Surfaces for Arbitrary Accelerated Observers* (2005), Int.J.Mod.Phys. **D16**, 1149 (2007) (arXiv gr-qc/0501090).
 - [26] Ashtekar A., *Asymptotic Structure of the Gravitational Field at Spatial Infinity*, in *General Relativity and Gravitation*, Vol.2, ed.Held A. (Plenum, New York, 1980).
 - [27] L.Lusanna, *The N- and 1-Time Classical Descriptions of N-Body Relativistic Kinematics and the Electromagnetic Interaction*, Int.J.Mod.Phys. **A12**, 645 (1997); *The Chronogeometrical Structure of Special and General Relativity: towards a Background-Independent Description of the Gravitational Field and Elementary Particles* (2004), in *General Relativity Research Trends*, ed. A.Reiner, Horizon in World Physics vol. 249 (Nova Science, New York, 2005) (gr-qc/0404122); *Towards a Unified Description of the Four Interactions in Terms of Dirac-*

- Bergmann Observables*, invited contribution to the book *Quantum Field Theory: a 20th Century Profile* of the Indian National Science Academy, ed. A.N.Mitra, foreward F.J.Dyson (Hindustan Book Agency, New Delhi, 2000) (hep-th/9907081).
- [28] P.A.M. Dirac, *Gauge Invariant Formulation of Quantum Electrodynamics*, Can.J.Phys. **33**, 650-659 (1955).
 - [29] S.Shanmugadhasan, *Canonical Formalism for Degenerate Lagrangians*, J.Math.Phys. **14**, 677 (1973).
 - [30] L.Lusanna, *The Shanmugadhasan Canonical Transformation, Function Groups and the Second Noether Theorem*, Int.J.Mod.Phys. **A8**, 4193 (1993); *The Relevance of Canonical Transformations in Gauge Theories and General Relativity*, Lecture Notes of "Seminario Interdisciplinare di Matematica" (Basilicata Univ.) **5**, 125 (2006).
 - [31] L.Lusanna and M.Pauri, *The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity - I: Dynamical Synchronization and Generalized Inertial Effects; II : Dirac versus Bergmann Observables and the Objectivity of Space-Time*, Gen.Rel.Grav. **38**, 187 and 229 (2006) (gr-qc/0403081 and 0407007). *Explaining Leibniz Equivalence as Difference of Non-Inertial Appearances: Dis-solution of the Hole Argument and Physical Individuation of Point-Events*, talk at the Oxford Conference on Spacetime Theory (2004), *Studies in History and Philosophy of Modern Physics* **37**, 692 (2006) (gr-qc/0604087). *Dynamical Emergence of Instantaneous 3-Spaces in a Class of Models of General Relativity*, in the book *Relativity and the Dimensionality of the World*, ed. A. van der Merwe, Springer Series *Fundamental Theories of Physics* **157**, 229 (2007) (gr-qc/0611045).
 - [32] S.B.Edgar and G.Ludwig, *Integration in the GHP Formalism IV: A New Lie Derivative Operator Leading to an Efficient Treatment of Killing Vectors*, Gen.Rel.Grav. **32**, 637 (2000).